J Math Chem (2014) 52:283–300
DOI 10.1007/s10910-013-0262-4

ORIGINAL PAPER

Asymptotics ($p \to \infty$) of L_p-norms of hypergeometric orthogonal polynomials

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Received: 2 May 2013 / Accepted: 7 September 2013 / Published online: 20 September 2013

Abstract The determination of the weighted L_p norms of the real orthogonal polynomials of hypergeometric type $\{y_n(x)\}$ is not only a very important problem *per se* in the theory of special functions, but also because of their recent entropic characterization and applications in quantum chemistry, quantum physics and information theory. Indeed, they essentially describe the pth-order Rényi and Tsallis entropies of the numerous quantum systems whose wavefunctions are controlled by these polynomials. Moreover, for different values of p, up to a constant factor, these norms characterize various fundamental and experimentally accessible quantities of many-electron systems. As well, the L_p norms have been used to develop and interpret all energy components in the density-functional theory of the ground-state of atoms and molecules. The asymptotics of these quantities when $n \to \infty$ and $p > 0$ have been recently calculated for Hermite polynomials, although not yet for Laguerre and Jacobi

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polynomials. Here, we determine the asymptotics ($p \to \infty$, n fixed) of the weighted L_p norms for general orthogonal polynomials in terms of the weight function and the coefficients of the second-order hypergeometric differential equation that they satisfy, and we apply it to the three classical families of real orthogonal polynomials. Moreover we analyse and discuss the monotonicity of this asymptotics, and we carry out a detailed numerical study of it.

Keywords Orthogonal polynomials · Quantum chemistry · Quantum physics · Information theory · Hermite polynomials · Laguerre polynomials · Jacobi polynomials · L_p-norms asymptotics

1 Introduction

Let $\{y_n(x)\}$ denote a sequence of real polynomials orthogonal with respect to the weight function $\omega(x)$ on the interval Δ. The probability density of the polynomial $y_n(x)$, to be called Rakhmanov's density heretoforth because he found [1] that it governs the asymptotic ($n \to \infty$) behaviour of the ratio y_{n+1}/y_n for general $\omega > 0$ almost everywhere on the finite interval Δ, is given by

$$\rho_n(x) = \frac{1}{d_n^2}\omega(x)y_n^2(x),$$

where d_n^2 is the normalization constant. Physically, $\rho_n(x)$ describes the probability density of the ground and excited states of the physical systems whose non-relativistic quantum-mechanical wavefunctions are controlled by the polynomials $y_n(x)$ (see e.g., [2,3]).

This density distribution can be characterized under certain conditions either by means of the ordinary moments $\mu_k := \int_\Delta x^k \rho_n(x)dx$ [4,5] with integer order k, or via the frequency moments (also called probability moments or entropic moments) of integer order k, $\nu_k := \int_\Delta [\rho_n(x)]^k\, dx$ [5–8]. The latter quantities are the integer-order instances of the weighted L_p norms of the orthonormal polynomials $y_n(x)$, which are set by

$$\|\rho_n\|_p \equiv \left(\int\limits_\Delta [\rho_n(x)]^p\, dx\right)^{\frac{1}{p}} = \frac{1}{d_n^2}\left(\int\limits_\Delta \left[\omega(x)y_n^2(x)\right]^p dx\right)^{\frac{1}{p}}\ ;\ p > 0. \quad (1)$$

They are closely related to not only the pth-order frequency moments $W_p[\rho_n] = \|\rho_n\|_p^p$, but also to various information-theoretic measures such as the Rényi entropies [9] $R_p[\rho_n]$, the Tsallis entropies [10], $T_p[\rho_n]$ and the Rényi spreading lengths [11] $L_p^R[\rho_n]$, which are defined as P

$$R_p[\rho_n] = \frac{1}{1-p}\ln W_p[\rho_n];\ \ p > 0, p \neq 1,$$

$$T_p[\rho_n] = \frac{1}{p-1}\left(1 - W_p[\rho_n]\right); \;\; p > 0, p \neq 1,$$
$$L_p^R[\rho_n] = \exp\left(R_p[\rho_n]\right) = \|\rho_n\|_p^{-\frac{p}{p-1}}.$$

respectively. These quantities, which include the Shannon information entropy $S[\rho_n] = -\int_\Delta \rho_n \ln \rho_n dx$ in the limiting case $q \to 1$, grasp different aspects of the distribution of the probability density $\rho_n(x)$ along the interval Δ when the order p is varying (see the recent review [12]).

From a chemical point of view these quantities do not only allow us to grasp various aspects of the internal disorder of many-electron systems (which are closely connected with the rich three-dimensional geometry of their electron distributions), but also they describe up to some constant factors numerous fundamental and experimentally accessible chemical quantities (average electron density, electron-nucleus attraction energy, kinetic and exchange energies, among many others; see e.g., [12–14]) and tightly bound other macroscopic properties of these systems [15–17]. Moreover, it most important that the L_p norms have been used to develop and interpret all energy components in the density-functional theory of the ground-state of atoms and molecules [18–20]. In addition, they have been used as uncertainty measures in chemical and physical quantum systems [21]. For further details, see the recent survey [12].

The analytical determination of these norms has been a long standing problem not only in quantum chemistry but also in the theory of special functions and extremal polynomials itself since the times of Bernstein and Steklov (see e.g., [22–24]) and in the theory of trigonometric series [8]. Recently, they have been calculated for polynomials $y_n(x)$ with arbitrary degree n by means of the combinatorics-based Bell polynomials in the Hermite [25], Laguerre [26] and Jacobi [27] cases; see also [28]. However, this methodology is computationally very demanding and analytically inefficient for high and very high values of n. Recently, extending some previous works of Aptekarev et al. [29] when $p \in \left[0, \frac{4}{3}\right]$, the strong asymptotics ($n \to \infty$ and $p > 0$) of the weighted L_p norms has been fully determined for Hermite polynomials $H_n(x)$ [30] orthogonal with respect to the weight function $\omega_H = e^{-x^2}$. This result has been achieved by means of the Tulyakov method [31] whose initial starting point is the recurrence relation of the polynomials; so, opposite to the matrix Riemann–Hilbert method [28] which begins with the orthogonality weight. We should immediately underline that both weak* [32] and strong [29] asymptotics of Laguerre and Jacobi polynomials are still lacking in spite of some serious efforts [23,33,34].

For completeness, it is worth mentioning here that the pth-power of the (non-weighted) L_p norms defined as $N_n(p;\omega) := \int_\Delta [y_n(x)]^p\, \omega(x)\, dx$ has also been considered and its asymptotics has been determined for $n \to \infty$ in some special cases [29,33,35]. Indeed, the leading term of the asymptotic behavior of the norms $N_n(p;\omega)$ for the polynomials orthogonal with respect to a weight function satisfying either the Bernstein condition or the weaker Szego condition has been calculated, obtaining more refined results for Jacobi polynomials [29,33]. As well, the asymptotic behavior of the Hermite norms $N_n(p;\omega_H^{1/2})$ when $n \to \infty$ has been found and applied by Larsson-Cohn [35] to some extremal problems on Wiener chaos [35]. In the discrete case, the only results found in the literature are the ones of Meixner [36] and Charlier [37] polynomials, which were recently used to determine the asymptotics of generalised derangements.

The aim of this work is the calculation of the complementary asymptotics (i.e., for $p \to \infty$ and n fixed) of the weighted L_p norms of general hypergeometric orthogonal polynomials (i.e., with respect to a general weight function on the real line) and their applications to the systems of orthogonal polynomials of Hermite, Laguerre and Jacobi types. We use the Laplace's asymptotic method, which is much more efficient for this problem than the ones based on linearisation [38–48], combinatorial [49] and integro-differential [50] techniques.

The structure of the paper is the following. In Sect. 2 we determine the asymptotics of the pth-power of the weigthed L_p norms of general orthogonal polynomials for large values of p by means of the Laplace's method [51] together with the second-order hypergeometric differential equation of the polynomials [52]. Then, in Sect. 3, we apply the previous result to Hermite, Laguerre and Jacobi polynomials. In Sect. 4 we analyse the monotonicity behaviour of the asymptotics of the L_p norms of these hypergeometric polynomials. Later, in Sect. 5, we make a numerical study of the weighted L_p norms in some special cases. Finally some conclusions and references are given.

2 Weighted L_p-norms of general orthogonal polynomials: asymptotics $(p \to \infty)$

In this section we determine the asymptotics ($p \to \infty$) of weighted L_p norms of the system of polynomials $\{y_n(x)\}$ orthogonal with respect to the weight function $\omega(x)$ on the interval (a, b) of the real line, not necessarily finite. According to Eq. (1) we have that the pth-power of the weighted L_p norm of the polynomials $y_n(x)$ is given by

$$\int_a^b \left[\omega(x) y_n^2(x)\right]^p dx = \int_a^b e^{p\left[\ln\omega(x)+\ln y_n^2(x)\right]} dx. \tag{2}$$

To estimate the asymptotic behaviour of this quantity when $p \to \infty$, we use the Laplace's method [51], designed to derive an asymptotic expansion of the functional integral

$$F[f] = \int_a^b e^{pf(x)} dx, \quad p > 0, \quad f(x) \text{ real}.$$

Let us suppose that this integral converges absolutely for large enough p. For large p, the dominant contribution of the integrand to the integral occurs around the point $x_0 \in [a, b]$ where $f(x)$ reaches its maximum value, so that the contribution of the integrand to the integral is exponentially damped away from it.

Assume that $f \in \mathcal{C}^3(a, b)$ and has only one simple maximum at $x = x_0$ in (a, b). Then $f'(x_0) = 0$, $f''(x_0) < 0$ and

$$F[f] = e^{pf(x_0)} \left[\sqrt{\frac{2\pi}{-pf''(x_0)}} + \mathcal{O}(p^{-1}) \right], \quad p \to \infty. \tag{3}$$

Moreover, if the function $f(x)$ has two or more maxima the dominat contribution to the functional $F[f]$ comes from the absolute maximum x_0 of $f(x)$, mainly because the contribution from a local maximum at x_1 is suppressed by the factor $\exp(f(x_1) - f(x_0))$. Then, the asymptotics of the weighted L_p norm of $y_n(x)$ is basically controlled by the absolute maximum of the function

$$f(x) = \ln \omega(x) + \ln y_n^2(x),$$

whose value $x_0 = x_0(n)$ is the solution of the equation

$$\frac{y_n'(x_0)}{y_n(x_0)} = -\frac{1}{2} \frac{\omega'(x_0)}{\omega(x_0)}. \tag{4}$$

So, from Eqs. (2)–(4) one has that

$$\int_a^b \left[\omega(x) y_n^2(x)\right]^p dx = \left[\omega(x_0) y_n^2(x_0)\right]^p \left[\sqrt{\frac{2\pi}{-pf''(x_0)}} + \mathcal{O}(p^{-1}) \right], p \to \infty, \tag{5}$$

where

$$f''(x_0) = \frac{\omega''(x_0)}{\omega(x_0)} - \frac{3}{2} \left[\frac{\omega'(x_0)}{\omega(x_0)}\right]^2 + \frac{2y_n''(x_0)}{y_n(x_0)}.$$

Furthermore, since the polynomial $y_n(x)$ satisfies [52] the hypergeometric differential equation

$$\sigma(x) y_n''(x) + \tau(x) y_n'(x) + \lambda_n y_n(x) = 0 \tag{6}$$

(where σ and τ are polynomials of degree 2 and 1, at most, respectively, and λ_n is a scalar), one has that

$$f''(x_0) = \frac{\omega''(x_0)}{\omega(x_0)} - \frac{3}{2} \left[\frac{\omega'(x_0)}{\omega(x_0)}\right]^2 - \frac{2\lambda_n}{\sigma(x_0)} + \frac{\tau(x_0)}{\sigma(x_0)} \frac{\omega'(x_0)}{\omega(x_0)}, \tag{7}$$

where use of the Laplace's condition (4) has been used.

3 Applications

In this section, we apply the general results (5) and (7) obtained in the previous section to determine the pth-power of the L_p norms of the three canonical systems of real orthogonal hypergeometric polynomials [52]; that is, the Hermite, Laguerre

Table 1 Data on classical orthogonal polynomials

	Hermite $H_n(x)$	Laguerre $L_n^{(\alpha)}(x)$ $(\alpha > -1)$	Jacobi $P_n^{(\alpha,\beta)}(x)$ $(\alpha > -1, \beta > -1)$
(a, b)	$(-\infty, +\infty)$	$(0, +\infty)$	$(-1, +1)$
$\omega(x)$	e^{-x^2}	$x^\alpha e^{-x}$	$(1-x)^\alpha(1+x)^\beta$
$\sigma(x)$	1	x	$1-x^2$
$\tau(x)$	$-2x$	$1+\alpha-x$	$-(\alpha+\beta+2)x+\beta-\alpha$
λ_n	$2n$	n	$n(n+\alpha+\beta+1)$
d_n^2	$2^n n!\sqrt{\pi}$	$\frac{\Gamma(n+\alpha+1)}{n!}$	$\frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}$

and Jacobi polynomials. These polynomials are known to satisfy the orthogonality condition

$$\int_a^b y_n(x)y_m(x)\omega(x)\,dx = d_n^2\delta_{mn},$$

and the second-order differential equation (6), whose coefficients $\sigma(x)$, $\tau(x)$, λ_n, weight function $\omega(x)$, orthogonality interval (a, b), and normalization constant d_n^2 are given in Table 1.

3.1 Hermite polynomials $H_n(x)$

These polynomials are orthogonal with respect to the weight function $\omega_H(x) = e^{-x^2}$ on the whole real line and satisfy the differential equation (6) with the coefficients (see Table 1).

$$\sigma(x) = 1, \quad \tau(x) = -2x, \quad \lambda_n = 2n.$$

According to (4), the absolute maximum $x_0 = x_0(n)$ is given by one solution of the equation

$$x_0 H_n(x_0) = 2n H_{n-1}(x_0).$$

Moreover, following Eq. (7) one has that the second derivative of $f_H(x) = \ln \omega_H(x) + \ln (H_n(x))^2$ evaluated at the absolute maximum x_0 has the value

$$f_H''(x_0) = 2x_0^2 - 4n - 2.$$

Then, according to Eq. (5) we obtain that

$$\int_{-\infty}^{+\infty}\left[\omega_H(x)H_n^2(x)\right]^p\,dx=2\int_0^{+\infty}\left[\omega_H(x)H_n^2(x)\right]^p\,dx$$
$$=2\left[\omega_H(x_0)H_n^2(x_0)\right]^p\left[\sqrt{\frac{2\pi}{p(4n-2x_0^2+2)}}+\mathcal{O}(p^{-1})\right]. \quad (8)$$

For the particular cases $n=0,1$ and 2 we have that $x_0=0,1$ and $\sqrt{\frac{5}{2}}$, respectively, so that the L_p norm of the corresponding polynomials has the asymptotical values

$$\sqrt{\frac{\pi}{p}},\ 2^{2p+1}e^{-p}\left[\sqrt{\frac{\pi}{2p}}+\mathcal{O}(p^{-1})\right]\quad\text{and}\quad 2^{6p+1}e^{-\frac{5}{2}p}\left[\sqrt{\frac{2\pi}{5p}}+\mathcal{O}(p^{-1})\right]. \quad (9)$$

Remark that the first of these three asymptotical values is the exact value of the functional. To calculate it we take into account that $x_0=0$, so that one has to apply the Laplace's method to the whole integration interval $(-\infty,+\infty)$ in (8).

3.2 Laguerre polynomials $L_n^{(\alpha)}(x)$

These polynomials are orthogonal with respect to the weight function $\omega_L(x)=x^\alpha e^{-x}$ on the interval $[0,+\infty)$ and satisfy the differential equation (6) with the coefficients

$$\sigma(x)=x,\quad \tau(x)=1+\alpha-x,\quad \lambda_n=n$$

Here, the absolute maximum $x_0=x_0(n)$ is according to Eq. (4) one solution of the equation

$$\left(\frac{\alpha}{x_0}-1\right)L_n^{(\alpha)}(x_0)=2L_{n-1}^{(\alpha+1)}(x_0). \quad (10)$$

Moreover, according to Eq. (7), one has that the second derivative of the function $f_L(x)=\ln\omega_L(x)+\ln\left(L_n^{(\alpha)}(x)\right)^2$ evaluated at x_0 has the value

$$f_L''(x_0)=\frac{\alpha^2}{2x_0^2}-\frac{2n+\alpha+1}{x_0}+\frac{1}{2}.$$

Then, following Eq. (5) we obtain that the pth-power of the weighted L_p norm of Laguerre polynomials $L_n^{(\alpha)}(x)$ has the following asymptotic behavior

$$\int_0^{+\infty} \left[\omega_L(x)\left[L_n^{(\alpha)}(x)\right]^2\right]^p dx = \left[\omega_L(x_0)\left[L_n^{(\alpha)}(x_0)\right]^2\right]^p \times \left[\sqrt{\frac{2\pi}{p\left(-\frac{\alpha^2}{2x_0^2}+\frac{2n+\alpha+1}{x_0}-\frac{1}{2}\right)}} + \mathcal{O}(p^{-1})\right], \tag{11}$$

for $p \to +\infty$ and $\alpha > 0$. For the Laguerre polynomials with $\alpha > 0$ and degrees $n = 0$ and 1, one finds from Eq. (10) the absolute maximum values $x_0 = \alpha$ and $\frac{1}{2}\left(2\alpha + 3 - \sqrt{8\alpha+9}\right)$, respectively. Taking these particular cases into Eq. (11), we obtain that the leading term of the asymptotics ($p \to \infty$) of the weighted L_p norms of the corresponding polynomials $L_0^{(\alpha)}(x) = 1$ and $L_1^{(\alpha)}(x) = \alpha + 1 - x$ is given by

$$\int_0^{+\infty} \left[\omega_L(x)\left[L_0^{(\alpha)}(x)\right]^2\right]^p dx = \alpha^{p\alpha} e^{-p\alpha}\left[\sqrt{\frac{2\pi\alpha}{p}} + \mathcal{O}(p^{-1})\right]$$

and

$$\int_0^{+\infty} \left[\omega_L(x)\left[L_1^{(\alpha)}(x)\right]^2\right]^p dx = \left[x_0^{\alpha} e^{-x_0}(1+\alpha-x_0)^2\right]^p \left[\sqrt{\frac{2\pi}{-pf_L''(x_0)}} + \mathcal{O}(p^{-1})\right],$$

respectively, with

$$f_L''(x_0) = \frac{3\sqrt{8\alpha+9} - 8\alpha - 9}{\left(\sqrt{8\alpha+9} - 2\alpha - 3\right)^2}.$$

Moreover, in the subcase $\alpha = 1$ one finds that $x_0 = \frac{1}{2}\left(5 - \sqrt{17}\right)$ and $f''(x_0) = -\frac{1}{16}\left(51 + 11\sqrt{17}\right)$, so that we obtain the following asymptotics

$$\int_0^{+\infty} \left[\omega_L(x)\left[L_1^{(1)}(x)\right]^2\right]^p dx = \left[\frac{1}{2}\left(31 - 7\sqrt{17}\right) e^{-\frac{1}{2}\left(5-\sqrt{17}\right)}\right]^p \times \left[\sqrt{\frac{32\pi}{p\left(51+11\sqrt{17}\right)}} + \mathcal{O}(p^{-1})\right].$$

3.3 Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$

These polynomials are orthogonal with respect to the weight function $\omega_J(x) = (1-x)^\alpha(1+x)^\beta$ on the interval $[-1,+1]$, and satisfy the differential equation (6) with the coefficients

$$\sigma(x) = 1 - x^2, \quad \tau(x) = \beta - \alpha - (\alpha+\beta+2)x, \quad \lambda_n = n(n+\alpha+\beta+1)$$

In this case the absolute maximum x_0 turns out to be, according to Eq. (4), one solution of the equation

$$\frac{P_{n-1}^{(\alpha+1,\beta+1)}(x_0)}{P_n^{(\alpha,\beta)}(x_0)} = -\frac{1}{\alpha+\beta+n+1}\left(\frac{-\alpha}{1-x_0} + \frac{\beta}{1+x_0}\right). \tag{12}$$

Moreover, the use of Eq. (7) allows us to find the following value for the second derivative of the function $f_J(x) = \ln\omega_J(x) + \ln\left(P_n^{(\alpha,\beta)}\right)^2$ evaluated at x_0:

$$\begin{aligned} f_J''(x_0) = &-\left(\alpha + \frac{\alpha^2}{2}\right)\frac{1}{(1-x_0)^2} - \left(\beta + \frac{\beta^2}{2}\right)\frac{1}{(1+x_0)^2} \\ &-\frac{\alpha\beta}{1-x_0^2} - \frac{2n(n+\alpha+\beta+1)}{1-x_0^2} \\ &+\frac{\beta-\alpha-(\alpha+\beta+2)x_0}{1-x_0^2}\left[\frac{\beta}{1+x_0} - \frac{\alpha}{1-x_0}\right]. \end{aligned} \tag{13}$$

Now, from Eq. (5) we find that the pth-power of the weighted L_p norm of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ has the following asymptotic behavior:

$$\begin{aligned} &\int_{-1}^{+1}\left[\omega_J(x)\left[P_n^{(\alpha,\beta)}(x)\right]^2\right]^p dx \\ &= \left[\omega_J(x_0)\left[P_n^{(\alpha,\beta)}(x_0)\right]^2\right]^p\left[\sqrt{\frac{2\pi}{-pf_J''(x_0)}} + \mathcal{O}(p^{-1})\right] \end{aligned} \tag{14}$$

for $p \to \infty$ and $\alpha, \beta > 0$. In the particular case where $n = 0$, $\alpha > 0$ and $\beta > 0$ we can find from Eq. (12) and (13) that

$$x_0 = \frac{\beta-\alpha}{\alpha+\beta} \quad \text{and} \quad f_J''(x_0) = -\frac{(\alpha+\beta)^3}{4\alpha\beta},$$

respectively. Thus, from Eq. (14) with these values of x_0 and $f_J''(x_0)$ we obtain the value

$$\int_{-1}^{+1} \left[\omega_J(x)\left(P_0^{(\alpha,\beta)}(x)\right)^2\right]^p dx$$
$$= 2^{p(\alpha+\beta)}\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha p}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta p}\left[\sqrt{\frac{8\pi\alpha\beta}{p(\alpha+\beta)^3}}+\mathcal{O}(p^{-1})\right]$$

for the leading term of the asymptotics ($p \to \infty$) of $P_0^{(\alpha,\beta)}(x) = 1$. Finally, let us consider another particular case: when $n = 1$ and $\alpha = \beta > 0$. Then, $P_1^{(\alpha,\alpha)}(x) = (1+\alpha)x$ and the associated Rakhmanov density

$$\rho_n^J(x) = \omega_J(x)\left[P_1^{(\alpha,\alpha)}\right]^2 = (1+\alpha)^2(1-x^2)^\alpha x^2$$

is symmetric with respect to the origin. Then, the corresponding pth-power of the weighted L_p norm can be expressed as

$$\int_{-1}^{+1}\left\{\omega_J(x)\left[P_1^{(\alpha,\alpha)}(x)\right]^2\right\}^p dx = 2\int_{0}^{+1}\left\{\omega_J(x)\left[P_1^{(\alpha,\alpha)}(x)\right]^2\right\}^p dx.$$

The application of the Laplace's method to this functional on the interval $(0, +1)$ together with the values

$$x_0 = \sqrt{\frac{1}{1+\alpha}} \quad \text{and} \quad f_J''(x_0) = -\frac{4(1+\alpha)^2}{\alpha}$$

for the absolute maximum and the second derivative of the corresponding function $f_J(x)$ allows us to find from Eq. (5) the expression

$$\int_{-1}^{+1}\left\{\omega_J(x)\left[P_1^{(\alpha,\alpha)}(x)\right]^2\right\}^p dx = 2(1+\alpha)^p\left[\frac{\alpha}{1+\alpha}\right]^{\alpha p}\left[\sqrt{\frac{\pi\alpha}{2(1+\alpha)^2 p}}+\mathcal{O}(p^{-1})\right]$$

for the leading term of the asymptotics ($p \to \infty$) of the weighted L_p norm of the Jacobi polynomial $P_1^{(\alpha,\alpha)}(x)$.

4 Monotonicity of the asymptotic behaviour

In this section we analyse the monotonicity of the asymptotics ($p \to +\infty$) for the Hermite polynomials $H_n(x)$. Then, we describe the difficulties to obtain the same property in the Laguerre and Jacobi cases.

We start from the general expression (5), whose asymptotic behaviour is given by

$$G(p) = \left[\omega(x_0) y_n^2(x_0)\right]^p \sqrt{\frac{2\pi}{-pf''(x_0)}}.$$

This function $G(p)$ is the product of the function $\left[\omega(x_0) y_n^2(x_0)\right]^p$ (which has, as a function of p, an exponential behavior in practice) and the power $p^{-\frac{1}{2}}$. For large values of p, the exponential function gives the increasing or decreasing behaviour of the asymptotics: $G(p)$ increases with p if $\omega(x_0) y_n^2(x_0) > 1$ and $G(p)$ decreases when p increases if $\omega(x_0) y_n^2(x_0) \leq 1$. Thus, notice that the monotonicity of the asymptotic behaviour for large values of p is controlled by the value of the function in the absolute maximum x_0.

Let us first consider the case of the orthogonal ($H_n(x)$) and orthonormal ($\tilde{H}_n(x)$) Hermite polynomials. Notice that $\tilde{H}_n(x) = H_n(x)/\sqrt{d_n^2}$. The orthonormal Hermite polynomials satisfy Eq. 18.14.9 of [53], namely

$$\omega_H(x)\tilde{H}_n^2(x) \leq \frac{1}{\sqrt{\pi}} < 1.$$

Thus, in particular we have that $\omega_H(x_0)\tilde{H}_n^2(x_0) < 1$, so the pth-power of the L_p norm of these polynomials decreases as p increases for large values of p.

On the other hand, the orthogonal Hermite polynomials, with the standard normalization constant d_n^2 given in Table 1, satisfy that [54]

$$\omega_H(x_0)H^2(x_0) > K_n,$$

for $n \geq 6$ with

$$K_n = \begin{cases} \frac{27}{61(2n)^{\frac{1}{6}}} \frac{2n\sqrt{4n-2}(n!)^2}{\sqrt{8n^2-8n+3}\left(\frac{n}{2}!\right)^2} & \text{if } n \text{ is even}, \\ \frac{27}{61(2n)^{\frac{1}{6}}} \frac{\sqrt{16n^2-16n+6}\,n!(n-1)!}{\sqrt{2n-1}\left(\frac{(n-1)}{2}!\right)^2} & \text{if } n \text{ is odd}. \end{cases}$$

Note that K_n reaches its minimum at $n = 6$, with value $K_6 \simeq 15209 > 1$. Then

$$\omega_H(x_0)H_n^2(x_0) > 1, \quad n \geq 6.$$

In fact this condition is also true for $1 \leq n \leq 5$. So finally we have that the pth-power of the L_p norm of these polynomials increases with p for large values of p. Moreover, for $n = 0$, $\omega_H(x_0)H_0^2(x_0) = 1$; so we have a decreasing behaviour in this case, as given by Eq. (9).

Let us now consider the Laguerre case. Then, the standard orthogonal $L_n^{(\alpha)}(x)$ and the orthornormal polynomials are related by $\tilde{L}_n^{(\alpha)}(x) = L_n^{(\alpha)}(x)/\sqrt{d_n^2}$. Regarding upper and lower bounds, we have not found in the literature results for these polynomials so simple and powerful as those for Hermite polynomials. However, notice that

$d_n^2 > 1$ for Laguerre polynomials with $\alpha > 0$. This implies that

$$\omega_L(x_0)\left(L_n^{(\alpha)}(x_0)\right)^2 > \omega_L(x_0)\frac{\left(L_n^{(\alpha)}(x_0)\right)^2}{d_n^2} = \omega_L(x_0)\left(\tilde{L}_n^{(\alpha)}(x_0)\right)^2. \tag{15}$$

Thus, if $\omega_L(x_0)\left(L_n^{(\alpha)}(x_0)\right)^2 < 1 \Rightarrow \omega_L(x_0)\left(\tilde{L}_n^{(\alpha)}(x_0)\right)^2 < 1$. Then, if the pth-power of the L_p norm of the standard orthogonal polynomials decreases as p increases for large values of p, the pth-power of the L_p norm of the orthonormal polynomials has the same behaviour.

For Jacobi polynomials, we do not have results like those for Hermite polynomials, neither their normalization constant is always greater or lower than one. Then, the monotonicity of the asymptotic behaviour remains open for each specific case.

5 Numerical study

In this section we study numerically the behaviour of the asymptotics given in Sect. 3 for the Hermite, Laguerre and Jacobi families of orthogonal polynomials. This is done in each case for the standard orthogonal and orthonormal polynomials. In all the following figures, we represent the numerically calculated exact value of the corresponding pth-power of the L_p norms $\|\rho\|_p^p$ of these polynomials (squared dot), its corresponding asymptotic behaviour γ_p (solid line), as well as the absolute difference (circled dot)

$$\theta_a = |\|\rho\|_p^p - \gamma_p|,$$

and the relative difference (times)

$$\theta_r = \frac{|\|\rho\|_p^p - \gamma_p|}{\|\rho\|_p^p},$$

as a function of p.

Figure 1 shows these values for the orthogonal Hermite polynomial with $n = 1$, $H_1(x)$. Herein we observe the increasing behaviour of the pth-power of the L_p norms and the asymptotic behaviour, as predicted in Sect. 4. The absolute difference increases with p indicating that the asymptotic behaviour can be improved with some increasing terms, while the relative difference goes to zero as expected for a good asymptotic behaviour.

In Fig. 2 we take into account the orthonormal Hermite polynomial $\tilde{H}_1(x)$, whose pth-power of the L_p norm is now decreasing, as expected from the results of Sect. 4. The absolute and relative differences naturally decrease as well.

These behaviours are completely analogous for Hermite polynomials with other values of the degree n.

Now, let us consider the orthogonal Laguerre polynomial with degree $n = 1$ and parameter $\alpha = 1$, $L_1^{(1)}(x)$. Figure 3 shows that the pth-power of the L_p norms

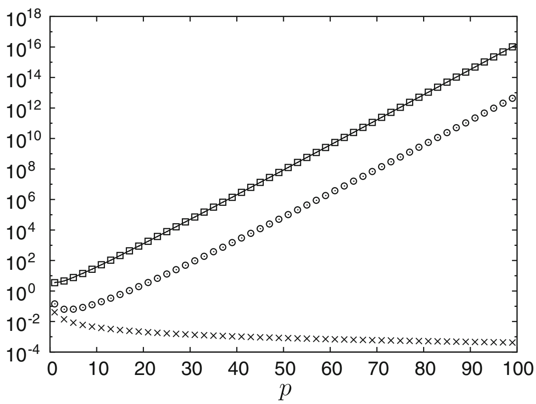


Fig. 1 Exact value of the pth-power of the L_p norms $\|\rho\|_p^p$ (*squared dot*), its corresponding asymptotic behaviour γ_p (*solid line*), absolute difference θ_a (*circled dot*), relative difference θ_r (*times*) for the Hermite polynomial $H_1(x)$, as a function of p

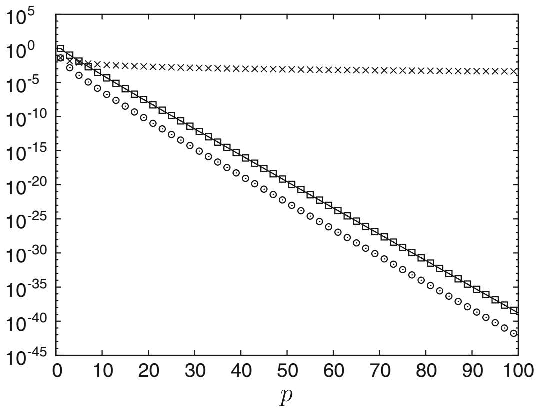


Fig. 2 Exact value of the pth-power of the L_p norms $\|\rho\|_p^p$ (*squared dot*), its corresponding asymptotic behaviour γ_p (*solid line*), absolute difference θ_a (*circled dot*), relative difference θ_r (*times*) for the orthonormal Hermite polynomial $\tilde{H}_1(x)$, as a function of p

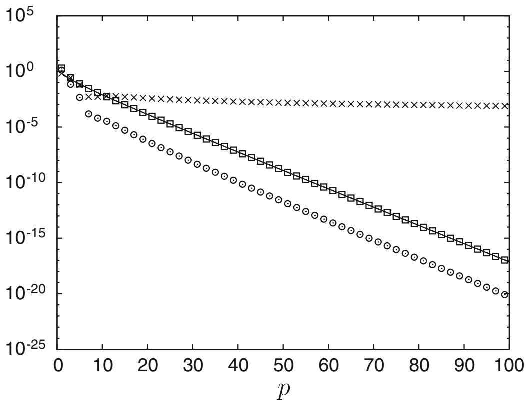


Fig. 3 Exact value of the pth-power of the L_p norms $\|\rho\|_p^p$ (*squared dot*), its corresponding asymptotic behaviour γ_p (*solid line*), absolute difference θ_a (*circled dot*), relative difference θ_r (*times*) for the Laguerre polynomial $L_1^{(1)}(x)$, as a function of p

decreases as a function of p. Then, taking into account the result (15) from Sect. 4, we can conclude that the pth-power of the L_p norms of the orthonormal polynomials $\tilde{L}_1^{(1)}(x)$ also decreases as p is increasing. The absolute and relative differences θ_a and θ_r, also decrease as p is increasing.

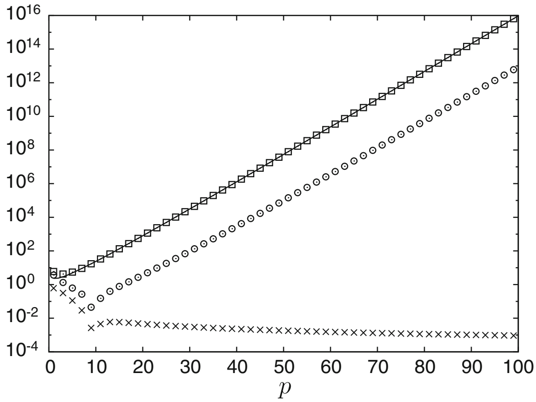


Fig. 4 Exact value of the pth-power of the L_p norms $\|\rho\|_p^p$ (*squared dot*), its corresponding asymptotic behaviour γ_p (*solid line*), absolute difference θ_a (*circled dot*), relative difference θ_r (*times*) for the Laguerre polynomial $L_1^{(2)}(x)$, as a function of p

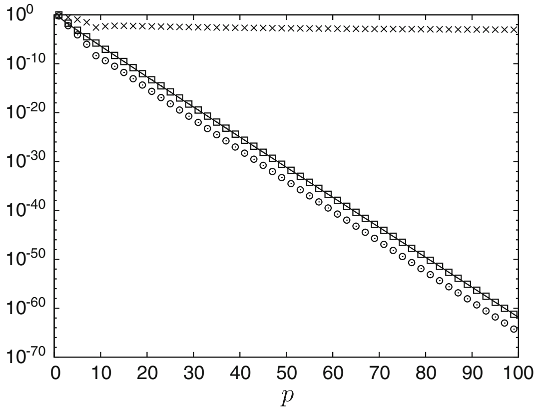


Fig. 5 Exact value of the pth-power of the L_p norms $\|\rho\|_p^p$ (*squared dot*), its corresponding asymptotic behaviour γ_p (*solid line*), absolute difference θ_a (*circled dot*), relative difference θ_r (*times*) for the orthonormal Laguerre polynomial $\tilde{L}_1^{(2)}(x)$, as a function of p

In Figs. 4 and 5 we consider the orthogonal $L_1^{(2)}(x)$ and orthonormal $\tilde{L}_1^{(2)}(x)$ Laguerre polynomials, respectively. Now, the pth-power of the L_p norm increases with p in the orthogonal case, but it decreases in the orthonormal case. A brief study of the corresponding functions $x^2e^{-x}\left(L_1^{(2)}(x)\right)^2$ and $x^2e^{-x}\left(\tilde{L}_1^{(2)}(x)\right)^2$ shows that their maximum values are greater and lower than 1, respectively. As in the previous Hermite case, the absolute difference θ_a increases with p for the orthogonal polynomial, indicating that the asymptotic behaviour can be improved with increasing terms.

The Jacobi polynomials, with bounded support, can have maximum values greater or lower than zero, depending on the values of the degree n and the parameters α and β, regardless if we are considering the orthogonal or the orthonormal version of the polynomials. Thus, Figs. 6 and 7 show the behaviour of the pth-power of the L_p norm for the polynomials $P_1^{(\frac{3}{2},\frac{3}{2})}(x)$ and $\tilde{P}_1^{(\frac{3}{2},\frac{3}{2})}(x)$, respectively.We notice that it

Fig. 6 Exact value of the pth-power of the L_p norms $\|\rho\|_p^p$ (*squared dot*), its corresponding asymptotic behaviour γ_p (*solid line*), absolute difference θ_a (*circled dot*), relative difference θ_r (*times*) for the Jacobi polynomial $P_1^{(\frac{3}{2},\frac{3}{2})}(x)$, as a function of p

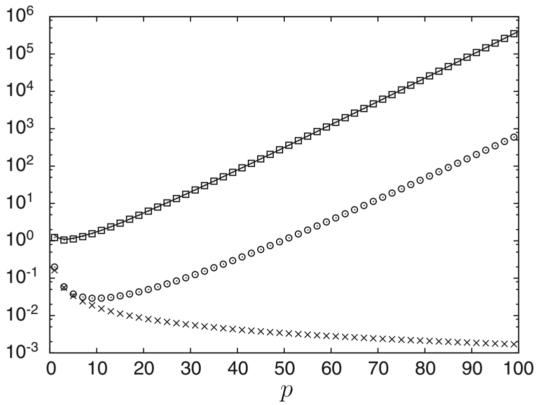
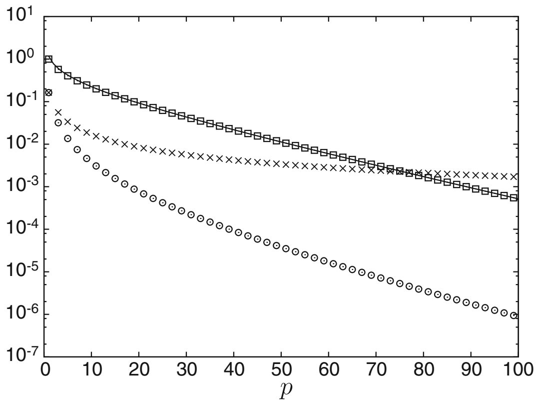


Fig. 7 Exact value of the pth-power of the L_p norms $\|\rho\|_p^p$ (*squared dot*), its corresponding asymptotic behaviour γ_p (*solid line*), absolute difference θ_a (*circled dot*), relative difference θ_r (*times*) for the orthonornal Jacobi polynomial $\tilde{P}_1^{(\frac{3}{2},\frac{3}{2})}(x)$, as a function of p



is increasing with p in the orthogonal case, but decreasing in the orthonormal case. According to the reasoning in Sect. 4, this behavior is because

$$\max_{x\in[-1,1]}\left\{(1-x)^{\frac{3}{2}}(1+x)^{\frac{3}{2}}\left(\tilde{P}_1^{(\frac{3}{2},\frac{3}{2})}(x)\right)^2\right\} < 1 < \max_{x\in[-1,1]}\left\{(1-x)^{\frac{3}{2}}(1+x)^{\frac{3}{2}}\left(P_1^{(\frac{3}{2},\frac{3}{2})}(x)\right)^2\right\}.$$

Figures 8 and 9 show the same quantities for polynomials $P_3^{(\frac{1}{2},\frac{1}{2})}(x)$ and $\tilde{P}_3^{(\frac{1}{2},\frac{1}{2})}(x)$, respectively. Now the behaviours are in the opposite direction compared with those of the previous examples in Figs. 6 and 7. The pth-power of the L_p norm is decreasing for the orthogonal polynomial but increasing for the orthonormal one. The reason, again, is that

Fig. 8 Exact value of the pth-power of the L_p norms $\|\rho\|_p^p$ (*squared dot*), its corresponding asymptotic behaviour γ_p (*solid line*), absolute difference θ_a (*circled dot*), relative difference θ_r (*times*) for the Jacobi polynomial $P_3^{(\frac{1}{2},\frac{1}{2})}(x)$, as a function of p

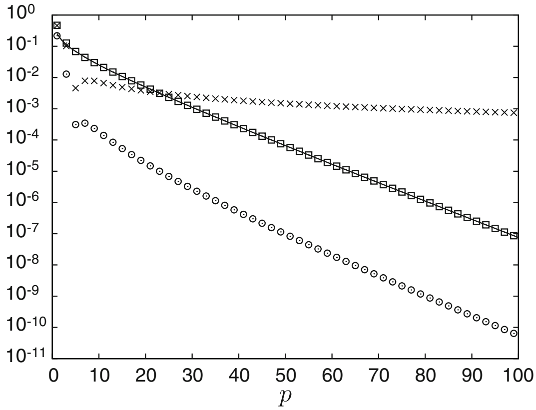
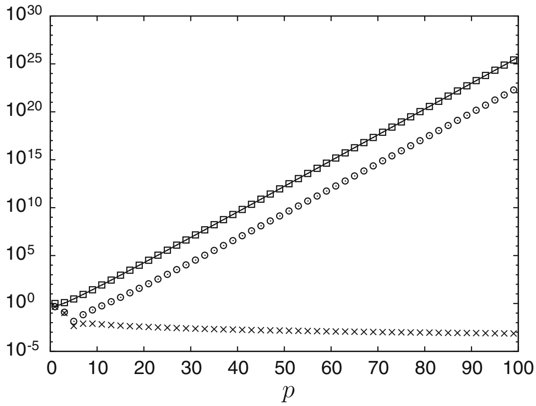


Fig. 9 Exact value of the pth-power of the L_p norms $\|\rho\|_p^p$ (*squared dot*), its corresponding asymptotic behaviour γ_p (*solid line*), absolute difference θ_a (*circled dot*), relative difference θ_r (*times*) for the orthonornal Jacobi polynomial $\tilde{P}_3^{(\frac{1}{2},\frac{1}{2})}(x)$, as a function of p



$$\max_{x\in[-1,1]}\left\{(1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}\left(P_3^{(\frac{1}{2},\frac{1}{2})}(x)\right)^2\right\} < 1 < \max_{x\in[-1,1]}\left\{(1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}\left(\tilde{P}_3^{(\frac{1}{2},\frac{1}{2})}(x)\right)^2\right\}.$$

Then, all the numerical examples considered here agree with the analytical monotonicity results of Sect. 4. Furthermore, the relative difference θ_r is always a decreasing function as p increases. Notice also in all the figures the exponential behavior predicted by the Laplace's method for the pth-power of the L_p norms and their asymptotics. Please keep in mind the logarithmic scale in the ordinate axis.

6 Conclusions and open problems

Energetic and entropic quantities of the ground and excited states of exactly and quasi-exactly solvable quantum systems can be often expressed in terms of some weighted L_p norms of the orthogonal polynomials which control the corresponding wavefunctions.

In this work we have determined the asymptotics of the weighted L_p norms of Hermite, Laguerre and Jacobi polynomials of nth degree when $p \to \infty$ by means of the Laplace's method. Moreover we have analyzed its monotonicity, identifying some new open problems. As well, a numerical study of the asymptotics for all classical continuous orthogonal polynomials has been performed.

The extension of these results to other continuous hypergeometric orthogonal polynomials of the Askey tableau [55] (even to those which are orthogonal with respect to a complex contour where the Laplace's method remains valid under certain conditions) and to the classical orthogonal polynomials in a discrete variable are open problems of a great interest in the theory of special functions not only from a fundamental point of view, but also because of their straightforward applications to various fields, from some weighted permutation problems to the quantum-mechanical description of physical systems.

Finally, let us point out that the only results found for the asymptotics of the L_p norms of discrete orthogonal polynomials up until now are the ones for the unweighted norms of Meixner [36] and Charlier [37] polynomials, which were shown to be very useful for some extremal problems in generalised derangements [36]. It is clear that for extensions to other discrete systems we will need the linearisation techniques for the corresponding polynomials [41,56–59].

Acknowledgments This work has been partially funded by the Junta-de-Andalucía Grants FQM-207, FQM-4643 and FQM-7276 as well as the MINECO Grant FIS2011-24540.

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